

NEW IDENTITIES IN DENDRIFORM ALGEBRAS

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ABSTRACT. Dendriform structures arise naturally in algebraic combinatorics (where they allow, for example, the splitting of the shuffle product into two pieces) and through Rota–Baxter algebra structures (the latter appear, among others, in differential systems and in the renormalization process of pQFT). We prove new combinatorial identities in dendriform dialgebras that appear to be strongly related to classical phenomena, such as the combinatorics of Lyndon words, rewriting rules in Lie algebras, or the fine structure of the Malvenuto–Reutenauer algebra. One of these identities is an abstract noncommutative, dendriform, generalization of the Bohnenblust–Spitzer identity and of an identity involving iterated Chen integrals due to C.S. Lam.

1. INTRODUCTION

Many associative algebras arising from combinatorial constructions have a nice property: their product can be splitted into two components that behave nicely with respect to the original product. The best known example of this phenomenon is given by Ree’s recursive definition and study of the shuffle product [Ree57]. The identities underlying this splitting, now called dendriform identities, can be traced back to the work of M.P. Schützenberger on the properties of Hall basis of free Lie algebras [Sch58]¹. However, in spite of Schützenberger’s seminal ideas, and of the regular use of the shuffle product –splitting–, e.g. in combinatorics or algebraic topology, dendriform structures were not investigated for their own till recently.

The situation has changed and, as explained below, dendriform algebras have risen a considerable interest. The purpose of the present article is to derive new dendriform identities and to study their applications to classical problems and structures in algebraic combinatorics, such as rewriting rules in free Lie algebras, properties of Lyndon words in relation with free Lie algebra basis, or Dynkin-type identities in the Malvenuto–Reutenauer algebra.

In abstract terms, a dendriform algebra is an algebra with left and right commuting representations on itself, written \prec and \succ , such that $x(y) = x \succ y$ and $(y)x = y \prec x$. The two actions add to form the product of the algebra. In the case of a commutative algebra, as an extra axiom, the left and right actions are further required to identify canonically (so that $x \succ y = y \prec x$, the particular case investigated in depth by Schützenberger in [Sch58]).

J.-L. Loday recently formalized this structure by introducing so-called dendriform identities in connection with dialgebra structures. Free dendriform algebras were described in terms of trees in [Lod01] (in fact, free commutative dendriform algebras had been described in [Sch58]). Following the work of M. Aguiar [Agu00], the first author of the present article constructed then a forgetful functor from associative Rota–Baxter algebras to dendriform algebras, as well as various forgetful functors from dendriform algebras to other types of algebras [E02]. Since Rota–Baxter algebras arise in many mathematical contexts (such as fluctuation theory, integral and finite differences calculus or perturbative renormalization in quantum field theory), this construction provides the theory of dendriform structures with a whole variety of new examples, besides the classical ones arising from shuffle algebras (such as the classical shuffle algebra or the algebras of singular cochains in algebraic topology).

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¹The third author thanks Ch. Reutenauer for communication of [Sch58].

This discovery was one of the leading motivation of the present article, that extends to the dendriform context ideas that have been developed by the authors, partly with J.M. Gracia-Bondía, in the setting of Rota–Baxter algebras [EGP07, EMP07], and that generalize to the noncommutative Rota–Baxter and dendriform setting classical results such as the Bohnenblust–Spitzer formula of fluctuation theory [Rota69] or Lam’s identities for iterated integrals and solutions of first order linear differential equations [Lam98].

Other results should be quoted here that have contributed to the development of the theory of dendriform structures. F. Chapoton [Chap02] (resp. M. Ronco [Ron02]) discovered that the classical proof of the Cartier–Milnor–Moore theorem [MM65] (respectively its modern combinatorial proof [Pat94]) could be extended to bialgebras with a dendriform structure, linking dendriform structures with other algebraic structures such as brace and pre-Lie algebras. Aguiar established in [Agu02] unexpected connections with the infinitesimal bialgebra structures studied in [Agu99, Agu01]. Another striking result in the field, and a great recent achievement in algebraic combinatorics, is due to L. Foissy, who was able to prove the Duchamp–Hivert–Thibon conjecture (the Lie algebra of primitive elements of the Malvenuto–Reutenauer Hopf algebra is a free Lie algebra) using another dendriform version of the Cartier–Milnor–Moore theorem [Foi05]. Other applications to algebraic combinatorics have been developed recently by F. Hivert, J. Novelli and Y. Thibon [NT06, HNT07]. These various results, together with the classical identities in free Lie algebras arising from the combinatorics of shuffles and of Hall and Lyndon basis, contributed strongly to motivate further the present article, and to the applications considered below of identities in dendriform algebras to questions in algebraic combinatorics.

2. OPERATIONS ON DENDRIFORM ALGEBRAS

In concrete terms, a *dendriform algebra* (or dendriform dialgebra) [Lod01] over a field k is a k -vector space A endowed with two bilinear operations \prec and \succ subject to the three axioms below:

- (1) $(a \prec b) \prec c = a \prec (b * c)$
- (2) $(a \succ b) \prec c = a \succ (b \prec c)$
- (3) $a \succ (b \succ c) = (a * b) \succ c,$

where $a * b$ stands for $a \prec b + a \succ b$. These axioms easily yield associativity for the law $*$. See [Sch58] for the commutative version, i.e. when furthermore $a \prec b = b \succ a$.

Example 1. *The shuffle dendriform algebra.* The tensor algebra $T(X)$ over an ordered alphabet is the linear span of the words (or noncommutative monomials) $y_1 \dots y_n$, $y_i \in X$ (we will also use, when convenient, the notation (y_1, \dots, y_n) for $y_1 \dots y_n$). The concatenation product on $T(X)$ is written by a dot: $y_1 \dots y_n \cdot z_1 \dots z_k := y_1 \dots y_n z_1 \dots z_k$. The tensor algebra is provided recursively with a dendriform algebra structure by the identities:

$$\begin{aligned} y_1 \dots y_n \prec z_1 \dots z_k &:= y_1(y_2 \dots y_n \prec z_1 \dots z_k + y_2 \dots y_n \succ z_1 \dots z_k) \\ y_1 \dots y_n \succ z_1 \dots z_k &:= z_1(y_1 \dots y_n \prec z_2 \dots z_k + y_1 \dots y_n \succ z_2 \dots z_k) \end{aligned}$$

Of course, this is nothing but a rewriting of Ree’s recursive definition of the shuffle product III, to which the associative product $\prec + \succ$ identifies [Ree57, Sch58].

Example 2. *The MAX dendriform algebra.* For any word w over the ordered alphabet X , let us write $\max(w)$ for the highest letter in w . The tensor algebra is provided with another dendriform algebra structure by the identities:

$$\begin{aligned} u \succ v &= u \cdot v \text{ if } \max(u) < \max(v) \text{ and } 0 \text{ else} \\ u \prec v &= u \cdot v \text{ if } \max(u) \geq \max(v) \text{ and } 0 \text{ else} \end{aligned}$$

where u and v run over the words over X . The associative product $\prec + \succ$ identifies with the concatenation product. MAX dendriform structures have appeared in the setting of noncommutative generalizations of the algebra of symmetric functions [NT06, HNT07].

Example 3. *The Malvenuto–Reutenauer dendriform algebra.* Let us write \mathbf{S}_* for the Malvenuto–Reutenauer algebra, that is, the direct sum of the group algebras of the symmetric groups $\mathbf{Q}[S_n]$, equipped with the (shifted) shuffle product (written $*$):

$$\forall(\sigma, \beta) \in S_n \times S_m, \sigma * \beta := (\sigma(1), \dots, \sigma(n)) \text{III}(\beta(1) + n, \dots, \beta(m) + n)$$

The restriction to \mathbf{S}_* of the dendriform structure on the tensor algebra provides \mathbf{S}_* with a dendriform algebra structure:

$$\begin{aligned} \sigma \prec \beta &:= \sigma(1) \cdot ((\sigma(2), \dots, \sigma(n)) \text{III}(\beta(1) + n, \dots, \beta(m) + n)) \\ \sigma \succ \beta &:= (\beta(1) + n) \cdot ((\sigma(1), \dots, \sigma(n)) \text{III}(\beta(2) + n, \dots, \beta(m) + n)). \end{aligned}$$

This structure is essentially the one used by Foissy to prove the Duchamp–Hivert–Thibon conjecture [Foi05].

Example 4. *Dendriform algebras of linear operators.* Let \mathcal{A} be any algebra of operator-valued functions on the real line, closed under integrals \int_0^x . One may wish to consider, for example, smooth $n \times n$ matrix-valued functions. Then, \mathcal{A} is a dendriform algebra for the operations:

$$A \prec B(x) := A(x) \cdot \int_0^x B(t) dt \quad A \succ B(x) := \int_0^x A(t) dt \cdot B(x)$$

with $A, B \in \mathcal{A}$. This is a particular example of a dendriform structure arising from a Rota–Baxter algebra structure. We refer to the last section of the article for further details on Rota–Baxter algebras and their connections to dendriform algebras. Here, let us simply mention that the Rota–Baxter operator on \mathcal{A} giving rise to the dendriform structure is: $R(A)(x) := \int_0^x A(t) dt$. Aguiar [Agu00] first mentioned the link between (weight zero) Rota–Baxter maps and dendriform algebras.

Besides the three products $\prec, \succ, *$, dendriform algebras carry naturally other operations. The most interesting, for our purposes, are the bilinear operations \triangleright and \triangleleft defined by:

$$(4) \quad a \triangleright b := a \succ b - b \prec a, \quad a \triangleleft b := a \prec b - b \succ a$$

that are left pre-Lie and right pre-Lie, respectively, which means that we have:

$$(5) \quad (a \triangleright b) \triangleright c - a \triangleright (b \triangleright c) = (b \triangleright a) \triangleright c - b \triangleright (a \triangleright c),$$

$$(6) \quad (a \triangleleft b) \triangleleft c - a \triangleleft (b \triangleleft c) = (a \triangleleft c) \triangleleft b - a \triangleleft (c \triangleleft b).$$

The associative operation $*$ and the pre-Lie operations $\triangleright, \triangleleft$ all define the same Lie bracket:

$$(7) \quad [a, b] := a * b - b * a = a \triangleright b - b \triangleright a = a \triangleleft b - b \triangleleft a.$$

$$\begin{array}{ccc} \text{dendriform alg.} & \xrightarrow{\triangleleft, \triangleright} & \text{pre-Lie alg.} \\ * \downarrow & & \downarrow [-, -] \\ \text{associative alg.} & \xrightarrow{[-, -]} & \text{Lie alg.} \end{array}$$

We recursively define on (A, \prec, \succ) , augmented by a unit 1:

$$a \prec 1 := a =: 1 \succ a \quad 1 \prec a := 0 =: a \succ 1,$$

implying $a * 1 = 1 * a = a$, the following set of elements for a fixed $x \in A$:

$$\begin{aligned} w_{\prec}^{(0)}(x) &= w_{\succ}^{(0)}(x) = 1, \\ w_{\prec}^{(n)}(x) &:= x \prec (w_{\prec}^{(n-1)}(x)), \\ w_{\succ}^{(n)}(x) &:= (w_{\succ}^{(n-1)}(x)) \succ x. \end{aligned}$$

Let us recall from Chapoton and Ronco [Chap02, Ron00, Ron02] that, in the free dendriform dialgebra on one generator a , augmented by a unit element, there is a Hopf algebra structure with respect to the associative product $*$. The elements $w_{\prec}^{(n)} := w_{\prec}^{(n)}(a)$ generate a cocommutative graded connected Hopf subalgebra $(H, *)$ with coproduct:

$$\Delta(w_{\prec}^{(n)}) = w_{\prec}^{(n)} \otimes 1 + 1 \otimes w_{\prec}^{(n)} + \sum_{0 < m < n} w_{\prec}^{(m)} \otimes w_{\prec}^{(n-m)},$$

and antipode $S(w_{\prec}^{(n)}) = (-1)^n w_{\prec}^{(n)}$. It is actually an easy exercise to check that the $w_{\prec}^{(n)}$ generate a free associative subalgebra of the free dendriform algebra on a for the $*$ product, so that one can use the previous formula for the coproduct action on $w_{\prec}^{(n)}$ as a definition of the Hopf algebra structure on H . As an important consequence, it follows that H is isomorphic, as a Hopf algebra, to the Hopf algebra of noncommutative symmetric functions [G+95]

We also define the following set of iterated left and right pre-Lie products (4). For $n > 0$, let $a_1, \dots, a_n \in A$:

$$(8) \quad \ell^{(n)}(a_1, \dots, a_n) := \left(\dots ((a_1 \triangleright a_2) \triangleright a_3) \dots \triangleright a_{n-1} \right) \triangleright a_n$$

$$(9) \quad r^{(n)}(a_1, \dots, a_n) := a_1 \triangleleft \left(a_2 \triangleleft (a_3 \triangleleft \dots (a_{n-1} \triangleleft a_n)) \dots \right).$$

For a fixed single element $a \in A$ we can write more compactly for $n > 0$:

$$(10) \quad \ell^{(n+1)}(a) = (\ell^{(n)}(a)) \triangleright a \quad \text{and} \quad r^{(n+1)}(a) = a \triangleleft (r^{(n)}(a))$$

and $\ell^{(1)}(a) := a =: r^{(1)}(a)$.

3. DENDRIFORM POWER SUMS EXPANSIONS

In the following we would like to address the theory of solutions of the following two equations for a fixed $a \in A$:

$$(11) \quad X = 1 + ta \prec X, \quad Y = 1 + Y \succ ta.$$

in $A[[t]]$. Formal solutions to these equations are given by the series of “left and right non associative power sums”:

$$X = \sum_{n \geq 0} t^n w_{\prec}^{(n)}(a) \quad \text{resp.} \quad Y = \sum_{n \geq 0} t^n w_{\succ}^{(n)}(a).$$

Notice that, due to the definition of the Hopf algebra structure on H , these two series behave as group-like elements with respect to coproduct Δ (up to the extension of the scalars from k to $k[t]$ and the natural extension of the Hopf algebra structure on $H = \bigoplus_{n \geq 0} H_n$ to its completion $\hat{H} = \prod_{n \geq 0} H_n$ with respect to the grading).

Recall now that the Dynkin operator is the linear endomorphism of the tensor algebra $T(X)$ over an alphabet $X = \{x_1, \dots, x_n, \dots\}$ into itself the action of which on words $y_1 \dots y_n$, $y_i \in X$ is given by the left-to-right iteration of the associated Lie bracket:

$$D(y_1, \dots, y_n) = [\dots [[y_1, y_2], y_3] \dots, y_n],$$

where $[x, y] := xy - yx$ [Reu93]. The Dynkin operator is a quasi-idempotent: its action on a homogeneous element of degree n satisfies $D^2 = nD$. The associated projector D/n sends $T_n(X)$, the component of degree n of the tensor algebra, to the component of degree n of the free Lie algebra over X . The tensor algebra is a graded connected cocommutative Hopf algebra, and it is natural to extend the definition of D to any

such Hopf algebra as the convolution product of the antipode S with the grading operator N : $D := S \star N$ [PR02, EGP06, EGP07, EMP07]. This applies in particular in the dendriform context to the Hopf algebra H introduced above. We will write D_n for $D \circ p_n$, where p_n is the canonical projection from $T(X)$ (resp. H) to $T_n(X)$ (resp. H_n).

Lemma 1. *For any integer $n \geq 1$ and for any $a \in A$ we have:*

$$(12) \quad D(w_{\prec}^{(n)}(a)) = \ell^{(n)}(a).$$

Proof. For $n = 1$ we have $D(w_{\prec}^{(1)}(a)) = D(a) = a = \ell^{(1)}(a)$. We then proceed by induction on n and compute:

$$\begin{aligned} D(w_{\prec}^{(n)}) &= (S \star N)(w_{\prec}^{(n)}) \\ &= \sum_{p=0}^{n-1} S(w_{\prec}^{(p)}) * N(w_{\prec}^{(n-p)}) \\ &= \sum_{p=0}^{n-1} S(w_{\prec}^{(p)}) * \left(N(w_{\prec}^{(n-p-1)}) \succ a \right) + \sum_{p=0}^{n-1} S(w_{\prec}^{(p)}) * \left((w_{\prec}^{(n-p-1)}) \succ a \right) \\ &= \sum_{p=0}^{n-1} S(w_{\prec}^{(p)}) * \left(N(w_{\prec}^{(n-p-1)}) \succ a \right) + (S \star \text{Id})(w_{\prec}^{(n)}) - S(w_{\prec}^{(n)}) \\ &= \sum_{p=0}^{n-1} S(w_{\prec}^{(p)}) * \left(N(w_{\prec}^{(n-p-1)}) \succ a \right) - S(w_{\prec}^{(n)}). \end{aligned}$$

Applying the identities:

$$(13) \quad x * (y \succ z) = (x * y) \succ z + x \prec (y \succ z),$$

$$(14) \quad S(w_{\prec}^{(n)}) = -a \prec S(w_{\prec}^{(n-1)}),$$

we get then:

$$\begin{aligned} D(w_{\prec}^{(n)}) &= \sum_{p=0}^{n-1} \left(S(w_{\prec}^{(p)}) * N(w_{\prec}^{(n-p-1)}) \right) \succ a + \sum_{p=1}^{n-1} S(w_{\prec}^{(p)}) \prec \left(N(w_{\prec}^{(n-p-1)}) \succ a \right) - S(w_{\prec}^{(n)}) \\ &= \left((S \star N)(w_{\prec}^{(n-1)}) \right) \succ a + \sum_{p=1}^{n-1} S(w_{\prec}^{(p)}) \prec \left(N(w_{\prec}^{(n-p-1)}) \succ a \right) - S(w_{\prec}^{(n)}) \\ &= D(w_{\prec}^{(n-1)}) \succ a - \sum_{p=1}^{n-1} \left(a \prec S(w_{\prec}^{(p-1)}) \right) \prec \left(N(w_{\prec}^{(n-p-1)}) \succ a \right) + a \prec S(w_{\prec}^{(n-1)}) \\ &= D(w_{\prec}^{(n-1)}) \succ a - \sum_{p=1}^{n-1} a \prec \left(S(w_{\prec}^{(p-1)}) * (N(w_{\prec}^{(n-p-1)}) \succ a) \right) + a \prec S(w_{\prec}^{(n-1)}) \\ &= D(w_{\prec}^{(n-1)}) \succ a - \sum_{p=1}^{n-1} a \prec \left(S(w_{\prec}^{(p-1)}) * (N - \text{Id})(w_{\prec}^{(n-p)}) \right) + a \prec S(w_{\prec}^{(n-1)}) \\ &= D(w_{\prec}^{(n-1)}) \succ a - \sum_{p=0}^{n-1} a \prec \left(S(w_{\prec}^{(p)}) * (N - \text{Id})(w_{\prec}^{(n-1-p)}) \right) \\ &= D(w_{\prec}^{(n-1)}) \succ a - a \prec \left((S \star (N - \text{Id}))(w_{\prec}^{(n-1)}) \right) \\ &= D(w_{\prec}^{(n-1)}) \succ a - a \prec D(w_{\prec}^{(n-1)}) \\ &= D(w_{\prec}^{(n-1)}) \triangleright a = \ell^{(n-1)}(a) \triangleright a = \ell^{(n)}(a). \end{aligned}$$

□

Theorem 2. [EGP07], [EMP07] *Let $H = \bigoplus_{n \geq 0} H_n$ be an arbitrary graded connected cocommutative Hopf algebra over a field of characteristic zero, and let again $\hat{H} = \prod_{n \geq 0} H_n$ be its completion with respect to the grading. The Dynkin operator $D \equiv S \star N$ induces a bijection between the group $G(H)$ of group-like elements of \hat{H} and the Lie algebra $\text{Prim}(H)$ of primitive elements in \hat{H} . The inverse morphism from $\text{Prim}(H)$ to $G(H)$ is given by*

$$(15) \quad h = \sum_{n \geq 0} h_n \mapsto \Gamma(h) := \sum_{n \geq 0} \sum_{\substack{i_1 + \dots + i_k = n \\ i_1, \dots, i_k > 0}} \frac{h_{i_1} \cdots h_{i_k}}{i_1(i_1 + i_2) \cdots (i_1 + \dots + i_k)}.$$

Since the element X (resp. Y) above is a group-like element in the Hopf algebra $\hat{H}[[t]]$, lemma 1 and theorem 2 imply the following two identities:

Theorem 3. *We have:*

$$(16) \quad w_{\succ}^{(n)}(a) = \sum_{\substack{i_1 + \dots + i_k = n \\ i_1, \dots, i_k > 0}} \frac{\ell^{(i_1)}(a) * \dots * \ell^{(i_k)}(a)}{i_1(i_1 + i_2) \cdots (i_1 + \dots + i_k)},$$

$$(17) \quad w_{\prec}^{(n)}(a) = \sum_{\substack{i_1 + \dots + i_k = n \\ i_1, \dots, i_k > 0}} \frac{r^{(i_k)}(a) * \dots * r^{(i_1)}(a)}{i_1(i_1 + i_2) \cdots (i_1 + \dots + i_k)}.$$

Proof. Identity (16) is indeed obtained immediately. Identity (17) can be derived easily as follows: consider the alternative dendriform structure on A defined by:

$$(18) \quad a \preceq b := -b \succ a, \quad a \succeq b := -b \prec a.$$

The associated associative algebra structure is then defined by:

$$(19) \quad a \overline{*} b := -b * a.$$

The two pre-Lie operations $\triangleright, \triangleleft$ are the same for both dendriform structures, and are related one to each other by:

$$(20) \quad a \triangleleft b = -b \triangleright a.$$

We can then obtain (17) from (16) and the identity:

$$w_{\prec}^{(n)}(a) = -w_{\succeq}^{(n)}(-a).$$

□

An alternative way to deduce (17) from (16) consists in applying the antipode S to both sides of (16): all $\ell^{(n)}(a)$'s are primitive, as we can see from the fact that $D(Y)$ is primitive and from applying lemma 1. The computation follows then easily by $S(\ell^{(n)}(a)) = -\ell^{(n)}(a) = (-1)^n r^{(n)}(a)$.

Example 5. Let us consider the MAX dendriform algebra $MAX(X)$ over a countable ordered alphabet $X = \{x_1, \dots, x_n, \dots\}$, (see example 2), and let us set $a := x_1 + \dots + x_n$. Then, we get immediately:

$$w_{\prec}^{(n)}(a) = x_1 \cdots x_n$$

whereas the multilinear part $m\ell^{(i)}(a)$ of $\ell^{(i)}(a)$ for $i \leq n$ (the component of $\ell^{(i)}(a)$ obtained by subtracting from $\ell^{(i)}(a)$ the monomials involving non trivial powers of the letters in X , so that e.g. $\ell^{(2)}(x_1 + x_2) = x_1x_2 - x_2x_1 - x_1^2 - x_2^2$ and $m\ell^{(2)}(x_1 + x_2) = x_1x_2 - x_2x_1$) is given by

$$m\ell^{(i)}(a) = \sum_{1 \leq j_1 < \dots < j_i \leq n} D(x_{j_1} \cdots x_{j_i}).$$

We will abbreviate $D(x_{j_1} \cdots x_{j_i})$ to $D(J)$, where $J = \{j_1, \dots, j_i\}$, so that:

$$ml^{(i)}(a) = \sum_{\substack{J \subset [n] \\ |J|=i}} D(J).$$

By theorem 3, we obtain (keeping only the multilinear part of the expansion on the right hand side):

$$x_1 \cdots x_n = \sum_{\substack{i_1 + \cdots + i_k = n \\ i_1, \dots, i_k > 0}} \sum_{\substack{J_1 \cdots J_k = [n] \\ |J_l| = i_l}} \frac{D(J_1) \cdots D(J_k)}{i_1(i_1 + i_2) \cdots (i_1 + \cdots + i_k)}.$$

Readers familiar with the Hopf algebraic approach to free Lie algebras advocated in [Reu93] will recognize that this identity may be rewritten as an expansion of the identity of $T(X)$ in terms of the Dynkin operator:

$$x_1 \cdots x_n = \sum_{\substack{i_1 + \cdots + i_k = n \\ i_1, \dots, i_k > 0}} \frac{D_{i_1} \star \cdots \star D_{i_k}}{i_1(i_1 + i_2) \cdots (i_1 + \cdots + i_k)} (x_1 \cdots x_n),$$

where \star stands for the convolution product in the set of linear endomorphisms of $T(X)$, $End(T(X))$ [Reu93, p.28].

Example 6. Let us turn to the Malvenuto–Reutenauer dendriform algebra. Here, we have: $w_{<}^{(n)}(1) = 1 \cdots n$, the identity in the symmetric group S_n . One can check that $r^{(n)}(1)$ is the image under the inversion in the symmetric group $\sigma \rightarrow I(\sigma) := \sigma^{-1}$ (extended linearly to the group algebra) of the iterated bracket: $[1, [2, \dots [n-1, n] \cdots]]$, with the usual convention: $[i, j] = ij - ji$. We get:

$$1 \cdots n = \sum_{\substack{i_1 + \cdots + i_k = n \\ i_1, \dots, i_k > 0}} \frac{I([1, \dots [i_k - 1, i_k] \cdots]) * \cdots * I([1, \dots [i_1 - 1, i_1] \cdots])}{i_1(i_1 + i_2) \cdots (i_1 + \cdots + i_k)}.$$

4. EXPONENTIAL EXPANSIONS OF DENDRIFORM POWER SUMS

The following describes an exponential expression of $Y = Y(t) = \sum_{n \geq 0} t^n w_{<}^{(n)}(a)$. An analogous result is readily derived for $X = X(t)$. Let us define the exponential map in terms of the associative product, $\exp^*(x) := \sum_{n \geq 0} x^{*n}/n!$. In $A[[t]]$ we may write the grading operator N naturally as $t\partial_t$.

Starting with the fact that $Y(t)$ is group-like in H we easily find in $A[[t]]$:

$$(21) \quad D(Y) = Y^{-1} * (t\dot{Y}),$$

hence $\dot{Y} = Y * \hat{\mathcal{L}}$, with $\hat{\mathcal{L}} := \hat{\mathcal{L}}(t) = \frac{D(Y)}{t} = \sum_{n \geq 0} \ell^{(n)}(a) t^{n-1}$. Using Magnus' expansion [Mag54] for the solution of first order linear differential equations, we immediately have $Y(t) = \exp^* \Omega(t)$, $\Omega(t) := \sum_{n \geq 0} \Omega^{(n)} t^n$, with:

$$(22) \quad \dot{\Omega}(t) = \frac{\text{ad } \Omega(t)}{1 - e^{-\text{ad } \Omega(t)}} \hat{\mathcal{L}}(t).$$

This leads to the following well known recursion for Ω :

$$(23) \quad \Omega(t) = \int_0^t \left(\hat{\mathcal{L}}(s) + \sum_{n \geq 0} (-1)^n \frac{B_n}{n!} [\text{ad}(\Omega(s))]^n (\hat{\mathcal{L}}(s)) \right) ds,$$

with B_n the Bernoulli numbers. For $n = 1, 2, 4$ we find $B_1 = -1/2$, $B_2 = 1/6$ and $B_4 = -1/30$, and $b_3 = b_5 = \cdots = 0$. For the first three terms in the Magnus expansion we find:

$$(24) \quad \Omega^{(1)} = \ell^{(1)}(a), \quad \Omega^{(2)} = \frac{1}{2} \ell^{(2)}(a), \quad \Omega^{(3)} = \frac{1}{3} \ell^{(3)}(a) + \frac{1}{12} [\ell^{(1)}(a), \ell^{(2)}(a)], \dots$$

We remark here that the Lie brackets can be written either in terms of the associative product $*$ or the left respectively right pre-Lie product.

5. LYNDON WORDS AND DENDRIFORM POWER SUMS

Now let a_1, \dots, a_n be a collection of elements in A . For any permutation $\sigma \in S_n$ we define the element $T_\sigma(a_1, \dots, a_n)$ as follows: define first the subset $E_\sigma \subset \{1, \dots, n\}$ by $k \in E_\sigma$ if and only if $\sigma_{k+1} > \sigma_j$ for any $j \leq k$, where we abbreviate $\sigma(i)$ to σ_i . We write E_σ in the increasing order:

$$1 \leq k_1 < \dots < k_p \leq n-1.$$

Then we set:

$$(25) \quad T_\sigma(a_1, \dots, a_n) := \ell^{(k_1)}(a_{\sigma_1}, \dots, a_{\sigma_{k_1}}) * \dots * \ell^{(n-k_p)}(a_{\sigma_{k_p+1}}, \dots, a_{\sigma_n})$$

Quite symmetrically we define the element $U_\sigma(a_1, \dots, a_n)$ by considering first the subset $F_\sigma \subset \{1, \dots, n\}$ defined by $l \in F_\sigma$ if and only if $\sigma_l < \sigma_j$ for any $j \geq l+1$. We write F_σ in the increasing order:

$$1 \leq l_1 < \dots < l_q \leq n-1.$$

Then we set:

$$(26) \quad U_\sigma(a_1, \dots, a_n) := r^{(l_1)}(a_{\sigma_1}, \dots, a_{\sigma_{l_1}}) * \dots * r^{(n-l_q)}(a_{\sigma_{l_q+1}}, \dots, a_{\sigma_n})$$

Following [Lam98] it is convenient to encode graphically the previous statistics on permutations. We write a permutation by putting a vertical bar (respectively a double bar) after each element of E_σ or F_σ according to the case. For example for the permutation $\sigma = (3261457)$ inside S_7 we have $E_\sigma = \{2, 6\}$ and $F_\sigma = \{4, 5, 6\}$. Putting the vertical bars:

$$(27) \quad \sigma = (32|6145|7), \quad \sigma = (3261||4||5||7)$$

we see that the corresponding elements in A will then be:

$$(28) \quad T_\sigma(a_1, \dots, a_7) = \ell^{(2)}(a_3, a_2) * \ell^{(4)}(a_6, a_1, a_4, a_5) * \ell^{(1)}(a_7)$$

$$(29) \quad = (a_3 \triangleright a_2) * \left(((a_6 \triangleright a_1) \triangleright a_4) \triangleright a_5 \right) * a_7,$$

$$(30) \quad U_\sigma(a_1, \dots, a_7) = r^{(4)}(a_3, a_2, a_6, a_1) * r^{(1)}(a_4) * r^{(1)}(a_5) * r^{(1)}(a_7)$$

$$(31) \quad = \left(a_3 \triangleleft (a_2 \triangleleft (a_6 \triangleleft a_1)) \right) * a_4 * a_5 * a_7.$$

Theorem 4. *For any a_1, \dots, a_n in the dendriform dialgebra A the following identities hold:*

$$(32) \quad \sum_{\sigma \in S_n} (\dots (a_{\sigma_1} \succ a_{\sigma_2}) \succ \dots) \succ a_{\sigma_n} = \sum_{\sigma \in S_n} T_\sigma(a_1, \dots, a_n),$$

$$(33) \quad \sum_{\sigma \in S_n} a_{\sigma_1} \prec (\dots (a_{\sigma_{n-1}} \prec a_{\sigma_n}) \dots) = \sum_{\sigma \in S_n} U_\sigma(a_1, \dots, a_n).$$

We postpone the proof to the next section, and first give some applications of the identities.

Example 7. Let us recall first the notion of Lyndon words. For a given ordered alphabet $X = \{x_1, \dots, x_n, \dots\}$, a Lyndon word is a word (an element $y_1 \dots y_n$, $y_i \in X$, of the free monoid X^* over X) that is strictly less in the lexicographical ordering than any of its proper right factors (i.e. strictly less than the $y_i \dots y_n$, $i > 1$). The length $lgt(w)$ of a word w is the number of letters (with repetitions) in w , so that e.g. $lgt(x_2 x_1 x_2 x_6) = 4$.

A fundamental theorem [Lot83] asserts that each word w in X^* has a unique Lyndon factorisation, i.e. can be written uniquely as

$$w = l_1 \dots l_k$$

where each l_i is a Lyndon word with $l_1 \geq \dots \geq l_k$. The sequence $(lgt(l_1), \dots, lgt(l_k))$ will be called the Lyndon sequence of w , and written $L(w)$. In the particular case where $X = [n]^{op} := \{n, n-1, \dots, 1\}$, the set of positive

integers with the decreasing ordering, it is easily checked that the Lyndon factorization of a permutation $\sigma \in S_n$, viewed as the word $\sigma(1) \cdots \sigma(n)$ over X is nothing but the decomposition introduced above in the definition of T_σ so that, for example, the Lyndon factorization of (3261457) is $32 \cdot 6145 \cdot 7$.

It is also well known that Lyndon words were first introduced to parameterize bases of the free Lie algebra [MR89]. This suggests that theorem 4 might be connected to properties of bases of free Lie algebras. This is indeed the case, and dendriform identities provide still another approach and contribution to their theory and the one of Lyndon words. This might seem not so surprising after all, since Schützenberger's discovery of the dendriform identities has been motivated by the construction of such bases. Notice however that the dendriform structure we use below is the MAX dendriform structure of example 2 and not the one classically used in the combinatorics of words—that is, the shuffle one. Notice also, that using different MAX dendriform structures on $T(X)$ (e.g. by reversing the order on X , and so on), would give rise to other combinatorial formulas than the ones obtained below.

So, let us consider $T(X)$, $X = \{x_1, \dots, x_n\}$, $x_i < x_{i+1}$ as a MAX dendriform algebra. Let β be an arbitrary permutation, and set: $a_1 := x_{\beta(1)}, \dots, a_n := x_{\beta(n)}$. We have:

$$\sum_{\sigma \in S_n} (\cdots (a_{\sigma(1)} \succ a_{\sigma(2)}) \succ \cdots) \succ a_{\sigma(n)} = (\cdots ((x_1 \succ x_2) \succ x_3) \cdots \succ x_n) = x_1 x_2 \cdots x_n,$$

since all the terms in the sum vanish, but one. On the other hand, for any words a, b in X^* without common letters, $a \triangleright b = [a, b]$ if $\max(a) < \max(b)$, and zero else. We get, for any sequence $(a_{i_1}, \dots, a_{i_k})$, with the i_j distinct:

$$(\cdots (a_{i_1} \triangleright a_{i_2}) \cdots \triangleright a_{i_k}) = 0$$

excepted if the sequence $\beta(i_1), \dots, \beta(i_k)$ is increasing, and then:

$$(\cdots (a_{i_1} \triangleright a_{i_2}) \cdots \triangleright a_{i_k}) = D(a_{i_1} \cdots a_{i_k}) = D(x_{\beta(i_1)} \cdots x_{\beta(i_k)}).$$

Let us write $Lyn(\beta)$ for the set of permutations $\sigma \in S_n$ such that, if $l_1(\sigma) \cdots l_{k(\sigma)}(\sigma)$, $l_i(\sigma) = \sigma(n_i) \cdots \sigma(n_{i+1} - 1)$ is the Lyndon factorization of σ (as defined above, that is, with respect to the *decreasing order on* $[n]$), then:

$$\forall i \leq k, \beta \circ \sigma(n_i) < \cdots < \beta \circ \sigma(n_{i+1} - 1).$$

Notice that, for $\sigma = 1 \dots n$, with Lyndon factorization of maximal length $1 \cdot 2 \cdot \cdots \cdot n$, we get $\sigma \in Lyn(\beta)$ for any $\beta \in S_n$.

For any sequence $S = (i_1, \dots, i_p)$ of elements of $[n]$, we write $D(S) := D(x_{i_1} \cdots x_{i_k})$. We also write $\beta(S)$ for $(\beta(i_1), \dots, \beta(i_p))$. We get, for any $\beta \in S_n$:

$$x_1 x_2 \cdots x_n = \sum_{\sigma \in Lyn(\beta)} D(\beta(l_1(\sigma))) \cdots D(\beta(l_{k(\sigma)}(\sigma))).$$

A point that should be noticed immediately is that this decomposition is *not* the classical decomposition of a word of X^* in the Lyndon basis, as described in [MR89] (and neither a direct variant thereof). The reason for this is that, by definition of the Dynkin operator, the opening brackets inside the blocks $D(\beta(l_j(\sigma)))$ are all set to the left, contrarily to what happens in the standard Lyndon factorizations defined in [MR89]. Since there is a unique permutation with Lyndon factorization of maximal length, $1 \dots n$, we also notice that, for any $\beta \in S_n$, this identity is a rewriting rule expanding $x_1 x_2 \cdots x_n$ as $x_{\beta(1)} \cdots x_{\beta(n)}$ plus a sum with integer coefficients of products of Lie brackets.

Let us consider a few examples. If $\sigma = 1 \dots n$, we have $Lyn(1 \dots n) = \{1 \dots n\}$, and the identity is trivial: $x_1 x_2 \cdots x_n = x_1 x_2 \cdots x_n$. If $\sigma = \omega = n \dots 1$, we get:

$$(34) \quad x_1 \cdots x_n = x_n \cdots x_1 + \sum_{\substack{s_1 \coprod \cdots \coprod s_k = [n] \\ s_i = \{n_1^i < \cdots < n_{|s_i|}^i\}}} \prod_{i=1 \dots k} D(n_1^i \cdots n_{|s_i|}^i)$$

where the sum runs over all the set partitions of $[n]$, ordered so that $\max(S_1) > \dots > \max(S_k)$, and where the last product is naturally ordered (the i -th term of the product is written to the left of the $(i+1)$ -th).

This decomposition has a striking property. For brevity sake, we refer the reader to [Reu93, Sect.5.6.2] for further details on the notions and results mentioned below. Recall that, for a given n , the Lie brackets $D(S)$, where S runs over all the words $S = 1S'$, S' a permutation of $\{2, \dots, n\}$ (e.g. $n = 5$, $S = 15234$, $S' = 5234$) form a basis (over any field of characteristic zero) of the multilinear part of the free Lie algebra on $[n]$ —let us call this basis the Dynkin basis. From this point of view, equation (34) gives nothing but the expansion of $x_1 \dots x_n$ in the Poincaré–Birkhoff–Witt (PBW) basis (see [Reu93, Th.0.2]) of the multilinear part of the free Lie algebra over X associated with the Dynkin basis.

For example, we get:

$$x_1 x_2 x_3 = x_3 x_2 x_1 + [x_2, x_3] x_1 + x_2 [x_1, x_3] + x_3 [x_1, x_2] + [[x_1, x_2], x_3],$$

$$\begin{aligned} x_1 x_2 x_3 x_4 = & x_4 x_3 x_2 x_1 + x_4 x_3 [x_1, x_2] + x_4 [x_2, x_3] x_1 + x_4 x_2 [x_1, x_3] + x_4 [[x_1, x_2], x_3] + [x_3, x_4] x_2 x_1 \\ & + [x_3, x_4] [x_1, x_2] + x_3 [x_2, x_4] x_1 + x_3 x_2 [x_1, x_4] + x_3 [[x_1, x_2], x_4] + [[x_2, x_3], x_4] x_1 \\ & + [x_2, x_4] [x_1, x_3] + x_2 [[x_1, x_3], x_4] + [[[x_1, x_2], x_3], x_4]. \end{aligned}$$

For a general σ , the expansion allows to rewrite σ as a sum of monomials of elements in the Dynkin basis. These results seem to be new, and connect the fine structure of free Lie algebras with our structural results on dendriform objects.

Example 8. Let us consider now the Malvenuto–Reutenauer dendriform algebra \mathbf{S}_* . Setting $a_1 = \dots = a_n := 1$ in theorem 4, we get:

$$\sum_{\sigma \in S_n} a_{\sigma_1} \prec (\dots (a_{\sigma_{n-1}} \prec a_{\sigma_n}) \dots) = n! \, 1 \dots n$$

On the other hand, we know that $r^{(n)}(1) = I([1, [\dots [n-1, n] \dots]])$ and get (using the symmetry between the definitions of T_σ and U_σ):

$$n! \, 1 \dots n = \sum_{\substack{i_1 + \dots + i_k = n, \\ i_j > 0}} |\{\sigma \in S_n, L(\sigma) = (i_1, \dots, i_k)\}| I([1, [\dots [i_k - 1, i_k] \dots]]) * \dots * I([1, [\dots [i_1 - 1, i_1] \dots]])$$

Since the Dynkin-type elements $I([1, [\dots [i_k - 1, i_k] \dots]])$ are algebraically independent in the Malvenuto–Reutenauer algebra (this follows e.g. from [G+95, Sect.5] and from the existence of an embedding of Solomon’s descent algebra in the Malvenuto–Reutenauer algebra [MR95]), one can identify the coefficients of the last sum with the corresponding coefficients of the expansion of $1 \dots n$ in example 2 in section 3. We get as a corollary an indirect (but conceptually interesting) computation of the number of permutations with a given Lyndon sequence:

$$|\{\sigma \in S_n, L(\sigma) = (i_1, \dots, i_k)\}| = \frac{n!}{i_1(i_1 + i_2) \dots (i_1 + \dots + i_k)}.$$

6. PROOF OF THE IDENTITY IN THEOREM 4

Notice that if the left-hand sides of (32) and (33) are by definition invariant under the permutation group S_n , it is not obvious at all that the right-hand sides share the same property. The proof of (32) proceeds by induction on the number n of arguments, and (33) will be easily deduced from (32). The case $n = 2$ reduces to the identity:

$$(35) \quad a_1 \succ a_2 + a_2 \succ a_1 = a_1 * a_2 + a_2 \triangleright a_1,$$

which immediately follows from the definitions. It is instructive to detail the case $n = 3$: considering the six permutations in S_3 :

$$(1|2|3), \quad (21|3), \quad (1|32), \quad (321), \quad (2|31), \quad (312),$$

we then compute, using axioms (2) and (3):

$$\begin{aligned} & a_1 * a_2 * a_3 + (a_2 \triangleright a_1) * a_3 + a_1 * (a_3 \triangleright a_2) + (a_3 \triangleright a_2) \triangleright a_1 + a_2 * (a_3 \triangleright a_1) + (a_3 \triangleright a_1) \triangleright a_2 \\ &= (a_1 \succ a_2 + a_2 \succ a_1) * a_3 + a_1 \succ (a_3 \triangleright a_2) + (a_3 \triangleright a_2) \succ a_1 + a_2 \succ (a_3 \triangleright a_1) + (a_3 \triangleright a_1) \succ a_2 \\ &= (a_1 \succ a_2) \succ a_3 + (a_2 \succ a_1) \succ a_3 + (a_1 \succ a_2) \prec a_3 + (a_2 \succ a_1) \prec a_3 \\ &\quad + a_1 \succ (a_3 \succ a_2) - a_1 \succ (a_2 \prec a_3) + (a_3 \succ a_2) \succ a_1 - (a_2 \prec a_3) \succ a_1 \\ &\quad + a_2 \succ (a_3 \succ a_1) - a_2 \succ (a_1 \prec a_3) + (a_3 \succ a_1) \succ a_2 - (a_1 \prec a_3) \succ a_2 \\ &= (a_1 \succ a_2) \succ a_3 + (a_2 \succ a_1) \succ a_3 + (a_1 \succ a_2) \prec a_3 + (a_2 \succ a_1) \prec a_3 \\ &\quad + (a_1 \succ a_3) \succ a_2 + (a_1 \prec a_3) \succ a_2 - a_1 \succ (a_2 \prec a_3) + (a_3 \succ a_2) \succ a_1 - (a_2 \prec a_3) \succ a_1 \\ &\quad + (a_2 \succ a_3) \succ a_1 + (a_2 \prec a_3) \succ a_1 - a_2 \succ (a_1 \prec a_3) + (a_3 \succ a_1) \succ a_2 - (a_1 \prec a_3) \succ a_2 \\ &= (a_1 \succ a_2) \succ a_3 + (a_2 \succ a_1) \succ a_3 + (a_1 \succ a_3) \succ a_2 \\ &\quad + (a_3 \succ a_1) \succ a_2 + (a_2 \succ a_3) \succ a_1 + (a_3 \succ a_2) \succ a_1. \end{aligned}$$

To start with the proof of the general case, we consider the following partition of the group S_n :

$$(36) \quad S_n = S_n^n \amalg \prod_{j,k=1}^{n-1} S_n^{j,k},$$

where S_n^n is the stabilizer of n in S_n , and where $S_n^{j,k}$ is the subset of those $\sigma \in S_n$ such that $\sigma_j = n$ and $\sigma_{j+1} = k$. We will set for $k \in \{1, \dots, n-1\}$:

$$(37) \quad S_n^k := \prod_{j=1}^{n-1} S_n^{j,k}.$$

This is the subset of permutations in S_n in which the two-terms subsequence (n, k) appears in some place. We have:

$$(38) \quad S_n = \prod_{j=1}^n S_n^j.$$

Each S_n^k is in bijective correspondence with S_{n-1} , in an obvious way for $k = n$, by considering the two-term subsequence (n, k) as a single letter for $k \neq n$. Precisely, in that case, in the expansion of $\sigma \in S_n$ as a sequence $(\sigma(1), \dots, \sigma(n))$, we replace the pair (n, k) by $n-1$ and any $j \in \{k+1, \dots, n-1\}$ by $j-1$, so that, for example, $(2, 1, 5, 3, 4) \in S_5^{3,3}$ is sent to $(2, 1, 4, 3)$. For each $\sigma \in S_n^k$ we denote by $\tilde{\sigma}$ its counterpart in S_{n-1} . Notice that for any $k \neq n$ and for any $j \in \{1, \dots, n-1\}$, the correspondence $\sigma \mapsto \tilde{\sigma}$ sends $S_n^{j,k}$ onto the subset of S_{n-1} formed by the permutations τ such that $\tau_j = n-1$. The following lemma is almost immediate:

Lemma 5. *For $\sigma \in S_n^n$ we have:*

$$(39) \quad T_\sigma(a_1, \dots, a_n) = T_{\tilde{\sigma}}(a_1, \dots, a_{n-1}) * a_n,$$

and for $\sigma \in S_n^k, k < n$ we have:

$$(40) \quad T_\sigma(a_1, \dots, a_n) = T_{\tilde{\sigma}}(a_1, \dots, \widehat{a_k}, \dots, a_{n-1}, a_n \triangleright a_k),$$

where a_k under the hat has been omitted.

We rewrite the $n - 1$ -term sequence $(a_1, \dots, \widehat{a_k}, \dots, a_{n-1}, a_n \triangleright a_k)$ as $(c_1^k, \dots, c_{n-1}^k)$. We are now ready to compute, using lemma 5 and the induction hypothesis:

$$\begin{aligned}
\sum_{\sigma \in S_n} T_\sigma(a_1, \dots, a_n) &= \sum_{k=1}^n \sum_{\sigma \in S_n^k} T_\sigma(a_1, \dots, a_n) \\
&= \sum_{\tau \in S_{n-1}} \left((\dots (a_{\tau_1} \succ a_{\tau_2}) \succ \dots) \succ a_{\tau_{n-1}} \right) * a_n + \sum_{k=1}^{n-1} \sum_{\tau \in S_{n-1}} \left((\dots (c_{\tau_1}^k \succ c_{\tau_2}^k) \succ \dots) \succ c_{\tau_{n-1}}^k \right) \\
&= \sum_{\tau \in S_{n-1}} \left((\dots (a_{\tau_1} \succ a_{\tau_2}) \succ \dots) \succ a_{\tau_{n-1}} \right) \succ a_n + \sum_{\tau \in S_{n-1}} \left((\dots (a_{\tau_1} \succ a_{\tau_2}) \succ \dots) \succ a_{\tau_{n-1}} \right) \prec a_n \\
&\quad + \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} \sum_{\substack{\tau \in S_{n-1} \\ \tau_j = n-1}} \left((\dots (\dots (c_{\tau_1}^k \prec c_{\tau_2}^k) \prec \dots (a_n \triangleright a_k)) \succ \dots) \succ c_{\tau_{n-1}}^k \right),
\end{aligned}$$

where $a_n \triangleright a_k = c_{\tau_j}^k = c_{n-1}^k$ lies in position j . Using the definition of the pre-Lie operation \triangleright and the axiom (3) we get:

$$\begin{aligned}
\sum_{\sigma \in S_n} T_\sigma(a_1, \dots, a_n) &= \sum_{\tau \in S_{n-1}} \left((\dots (a_{\tau_1} \succ a_{\tau_2}) \succ \dots) \succ a_{\tau_{n-1}} \right) \succ a_n + \sum_{\tau \in S_{n-1}} \left((\dots (a_{\tau_1} \succ a_{\tau_2}) \succ \dots) \succ a_{\tau_{n-1}} \right) \prec a_n \\
&\quad + \sum_{k=1}^{n-1} \sum_{\substack{\tau \in S_{n-1} \\ \tau_1 = n-1}} \left((\dots ((a_n \succ a_k) \succ c_{\tau_2}^k) \succ \dots) \succ c_{\tau_{n-1}}^k \right) - \sum_{k=1}^{n-1} \sum_{\substack{\tau \in S_{n-1} \\ \tau_1 = n-1}} \left((\dots ((a_n \prec a_k) \succ c_{\tau_2}^k) \succ \dots) \succ c_{\tau_{n-1}}^k \right) \\
&\quad + \sum_{k=1}^n \sum_{j=2}^{n-1} \sum_{\substack{\tau \in S_{n-1} \\ \tau_j = n-1}} \left((\dots \left((\dots (c_{\tau_1}^k \succ c_{\tau_2}^k) \succ \dots) \succ a_n \right) \succ a_k) \succ \dots \right) \succ c_{\tau_{n-1}}^k \\
&\quad + \sum_{k=1}^n \sum_{j=2}^{n-1} \sum_{\substack{\tau \in S_{n-1} \\ \tau_j = n-1}} \left((\dots \left((\dots (c_{\tau_1}^k \succ c_{\tau_2}^k) \succ \dots) \prec a_n \right) \succ a_k) \succ \dots \right) \succ c_{\tau_{n-1}}^k \\
&\quad - \sum_{k=1}^{n-1} \sum_{j=2}^{n-1} \sum_{\substack{\tau \in S_{n-1} \\ \tau_j = n-1}} \left((\dots (\dots (c_{\tau_1}^k \succ c_{\tau_2}^k) \succ \dots (a_k \prec a_n)) \succ \dots) \succ c_{\tau_{n-1}}^k \right),
\end{aligned}$$

where a_n lies in position j (resp. $j + 1$) in lines 4 and 5 (resp. in the last line) in the above computation, and where a_k lies in position $j + 1$ (resp. j) in lines 4 and 5 (resp. in the last line). We can rewrite this going back to the permutation group S_n and using the partition (36):

$$\begin{aligned}
\sum_{\sigma \in S_n} T_\sigma(a_1, \dots, a_n) &= \sum_{\sigma \in S_n^n} \left((\dots (a_{\sigma_1} \succ a_{\sigma_2}) \succ \dots) \succ a_{\sigma_{n-1}} \right) \succ a_{\sigma_n} \\
&\quad + \sum_{\sigma \in S_n^n} \left((\dots (a_{\sigma_1} \succ a_{\sigma_2}) \succ \dots) \succ a_{\sigma_{n-1}} \right) \prec a_{\sigma_n} \\
&\quad + \sum_{k=1}^{n-1} \sum_{\sigma \in S_n^{1,k}} \left((\dots ((a_{\sigma_1} \succ a_{\sigma_2}) \succ a_{\sigma_3}) \succ \dots) \succ a_{\sigma_n} \right) \\
&\quad - \sum_{k=1}^{n-1} \sum_{\sigma \in S_n^{1,k}} \left((\dots ((a_{\sigma_1} \prec a_{\sigma_2}) \succ a_{\sigma_3}) \succ \dots) \succ a_{\sigma_n} \right)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^n \sum_{j=2}^{n-1} \sum_{\sigma \in S_n^{j,k}} \left(\cdots \left(\left((\cdots (a_{\sigma_1} \succ a_{\sigma_2}) \succ \cdots) \succ a_{\sigma_j} \right) \succ a_{\sigma_{j+1}} \right) \succ \cdots \right) \succ a_{\sigma_n} \\
& + \sum_{k=1}^n \sum_{j=2}^{n-1} \sum_{\sigma \in S_n^{j,k}} \left(\cdots \left(\left((\cdots (a_{\sigma_1} \succ a_{\sigma_2}) \succ \cdots) \prec a_{\sigma_j} \right) \succ a_{\sigma_{j+1}} \right) \succ \cdots \right) \succ a_{\sigma_n} \\
& - \sum_{k=1}^{n-1} \sum_{j=2}^{n-1} \sum_{\sigma \in S_n^{j,k}} \left(\cdots \left((\cdots (a_{\sigma_1} \succ a_{\sigma_2}) \succ \cdots (a_{\sigma_{j+1}} \prec a_{\sigma_j})) \succ \cdots \right) \succ a_{\sigma_n} \right).
\end{aligned}$$

Lines 1,3 and 5 together give the left-hand side of (32) whereas lines 2, 4, 6 and 7 cancel. More precisely line 2 cancels with the partial sum corresponding to $j = n - 1$ in line 7, line 4 cancels with the partial sum corresponding to $j = 2$ in line 6, and (for $n \geq 4$), the partial sum corresponding to some fixed $j \in \{3, \dots, n-1\}$ in line 6 cancels with the partial sum corresponding to $j - 1$ in line 7. This proves equality (32).

We could prove *mutatis mutandis* (33) exactly along the same lines, but we can show that the two versions are in fact equivalent: The term $T_{\sigma}^{\succeq}(a_1, \dots, a_n)$ is defined the same way as $T_{\sigma}(a_1, \dots, a_n)$ has been defined before, but with the dendriform operation \succeq instead of \succ .

Lemma 6. *For any $\sigma \in S_n$ and for any $a_1, \dots, a_n \in A$ we have:*

$$(41) \quad U_{\sigma}(a_1, \dots, a_n) = (-1)^{n-1} T_{\omega\sigma\omega}^{\succeq}(a_n, \dots, a_1).$$

Proof. We denote by ω the permutation $(n \cdots 21)$ in S_n , and we set $b_j = a_{\omega_j}$, hence:

$$(b_1, \dots, b_n) := (a_n, \dots, a_1).$$

Using (19), (20) and the symmetry:

$$(42) \quad E_{\omega\sigma\omega} = n - F_{\sigma} \text{ for any } \sigma \in S_n,$$

we compute:

$$\begin{aligned}
U_{\sigma}(a_1, \dots, a_n) &= \left(a_{\sigma_1} \triangleleft (\cdots (a_{\sigma_{l_1-1}} \triangleleft a_{\sigma_{l_1}})) \cdots \right) * \cdots * \left(a_{\sigma_{k_q+1}} \triangleleft (\cdots (a_{\sigma_{n-1}} \triangleleft a_{\sigma_n})) \cdots \right) \\
&= (-1)^{n-1} \left(\cdots ((a_{\sigma_n} \triangleright a_{\sigma_{n-1}}) \triangleright \cdots) \triangleright a_{\sigma_{n-k_1-1}} \right) \bar{*} \cdots \bar{*} \left(\cdots ((a_{\sigma_{n-k_p}} \triangleright a_{\sigma_{n-k_p-1}}) \triangleright \cdots) \triangleright a_{\sigma_1} \right) \\
&= (-1)^{n-1} \left(\cdots ((a_{(\sigma\omega)_1} \triangleright a_{(\sigma\omega)_2}) \triangleright \cdots) \triangleright a_{(\sigma\omega)_{k_1}} \right) \bar{*} \cdots \bar{*} \left(\cdots ((a_{(\sigma\omega)_{k_p+1}} \triangleright a_{(\sigma\omega)_{k_p+2}}) \triangleright \cdots) \triangleright a_{(\sigma\omega)_n} \right) \\
&= (-1)^{n-1} \left(\cdots ((b_{(\omega\sigma\omega)_1} \triangleright b_{(\omega\sigma\omega)_2}) \triangleright \cdots) \triangleright b_{(\omega\sigma\omega)_{k_1}} \right) \bar{*} \cdots \bar{*} \left(\cdots ((b_{(\omega\sigma\omega)_{k_p+1}} \triangleright b_{(\omega\sigma\omega)_{k_p+2}}) \triangleright \cdots) \triangleright b_{(\omega\sigma\omega)_n} \right) \\
&= (-1)^{n-1} T_{\omega\sigma\omega}^{\succeq}(b_1, \dots, b_n).
\end{aligned}$$

□

Hence we compute, using successively the S_n -invariance, equation (32) and lemma 6:

$$\begin{aligned}
\sum_{S_n} a_{\sigma_1} \prec (\cdots \prec (a_{\sigma_{n-1}} \prec a_{\sigma_n}) \cdots) &= \sum_{S_n} b_{(\omega\sigma)_1} \prec (\cdots \prec (b_{(\omega\sigma)_{n-1}} \prec b_{(\omega\sigma)_n}) \cdots) \\
&= (-1)^{n-1} \sum_{S_n} (\cdots (b_{(\omega\sigma)_n} \succeq b_{(\omega\sigma)_{n-1}}) \succeq \cdots) \succeq b_{(\omega\sigma)_1} \\
&= (-1)^{n-1} \sum_{S_n} T_{\omega\sigma\omega}^{\succeq}(b_1, \dots, b_n) \\
&= \sum_{S_n} U_{\sigma}(a_1, \dots, a_n),
\end{aligned}$$

which finishes the proof of the theorem.

7. ROTA–BAXTER ALGEBRAS AND DENDRIFORM ALGEBRAS

Recall [E02] that an associative Rota–Baxter algebra (over a field k) is an associative algebra (A, \cdot) endowed with a linear map $R : A \rightarrow A$ subject to the following relation:

$$(43) \quad R(a)R(b) = R(R(a)b + aR(b) + \theta ab).$$

where $\theta \in k$. The map R is called a *Rota–Baxter operator of weight θ* . The map $\tilde{R} := -\theta \text{id} - R$ also is a weight θ Rota–Baxter map.

Proposition 7. [E02] *Any Rota–Baxter algebra gives rise to two dendriform dialgebra structures given by:*

$$(44) \quad a \prec b := aR(b) + \theta ab = -a\tilde{R}(b), \quad a \succ b := R(a)b,$$

$$(45) \quad a \prec' b := aR(b), \quad a \succ' b := R(a)b + \theta ab = -\tilde{R}(a)b.$$

The associated associative product $*$ is given for both structures by $a * b = aR(b) + R(a)b + \theta ab$. It is sometimes called the “double Rota–Baxter product”, and verifies:

$$(46) \quad R(a * b) = R(a)R(b),$$

which is just a reformulation of the Rota–Baxter relation (43).

Remark 8. [E02] *In fact, by splitting again the binary operation \prec (or alternatively \succ'), any Rota–Baxter algebra is tri-dendriform, in the sense that the Rota–Baxter structure yields three binary operations $<, \diamond, >$ subject to axioms refining the axioms of dendriform dialgebras [LR04]. The three binary operations are defined by $a < b = aR(b)$, $a \diamond b = \theta ab$ and $a > b = R(a)b$. Choosing to put the operation \diamond to the $<$ or $>$ side gives rise to the two dendriform structures above.*

Theorem 4 in the Rota–Baxter setting thus takes the following form:

Corollary 9. *Let (A, R) be a weight θ Rota–Baxter algebra, let $*$ be the double Rota–Baxter product defined above. Then, with the notations of section 2 we have:*

$$(47) \quad \sum_{\sigma \in S_n} R\left(\cdots R(R(a_{\sigma_1})a_{\sigma_2}) \cdots a_{\sigma_{n-1}}\right)a_{\sigma_n} = \sum_{\sigma \in S_n} T_{\sigma}(a_1, \dots, a_n),$$

$$(48) \quad \sum_{\sigma \in S_n} a_{\sigma_1} R\left(a_{\sigma_2} \cdots R(a_{\sigma_{n-1}} R(a_{\sigma_n})) \cdots\right) = \sum_{\sigma \in S_n} U'_{\sigma}(a_1, \dots, a_n),$$

where $U'_{\sigma}(a_1, \dots, a_n)$ is defined the same way as $U_{\sigma}(a_1, \dots, a_n)$ previously, but with the dendriform structure (A, \prec', \succ') . The pre-Lie operation \triangleright (resp. \triangleleft') involved in the right-hand side of equality (47) (resp. (48)) is given by:

$$(49) \quad a \triangleright b = R(a)b - bR(a) - \theta ba = [R(a), b] - \theta ba, \quad \text{resp.} \quad a \triangleleft' b = aR(b) - R(b)a - \theta ba = [a, R(b)] - \theta ba.$$

Applying the Rota–Baxter operator R to both sides of these two identities gives back the noncommutative Bohnenblust–Spitzer identity which has been announced in [EGP07] and proved in [EMP07] (Theorem 7.1). What we have obtained in theorem 4 is thus an extension of this noncommutative Bohnenblust–Spitzer identity to the dendriform setting.

In the weight $\theta = 0$ case, the pre-Lie operation reduces to $a \triangleright b = [R(a), b] = -b \triangleleft' a = -b \triangleleft a$. This case, in the form (48), has been handled by C.S. Lam in [Lam98], in the concrete situation when A is a function space on the real line, and when $R(f)$ is the primitive of f which vanishes at a fixed $T \in \mathbb{R}$. The formulation of theorem 4 in the general dendriform setting thus permits an application to Rota–Baxter operators of any weight θ .

In the particular case of a commutative Rota–Baxter algebra the identities in corollary 9 reduce to one since both Rota–Baxter pre-Lie products (49) agree. One recovers the classical Spitzer identity of fluctuation theory, and Rota’s generalization thereof to arbitrary commutative Rota–Baxter algebras [Rota69, RS72].

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